J Stat Phys (2008) 130: 535–544
DOI 10.1007/s10955-007-9453-3

Global Existence Proof for Relativistic Boltzmann Equation with Hard Interactions

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Received: 6 December 2005 / Accepted: 4 October 2007 / Published online: 16 October 2007

Abstract By combining the DiPerna and Lions techniques for the nonrelativistic Boltzmann equation and the Dudyński and Ekiel-Jeżewska device of the causality of the relativistic Boltzmann equation, it is shown that there exists a global mild solution to the Cauchy problem for the relativistic Boltzmann equation with the assumptions of the relativistic scattering cross section including some relativistic hard interactions and the initial data satisfying finite mass, energy and entropy. This is in fact an extension of the result of Dudyński and Ekiel-Jeżewska to the case of the relativistic Boltzmann equation with hard interactions.

Keywords Relativistic Boltzmann equation · Global existence · Mild solution

1 Introduction

We are concerned with a global existence of mild solution to the Cauchy problem for the relativistic Boltzmann equation with the relativistic scattering cross section including some relativistic hard interactions through initial data satisfying finite mass, energy and entropy. The relativistic Boltzmann equation (hereafter RBE) is of the following dimensionless form (see [5])

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{p_0}\frac{\partial f}{\partial \mathbf{x}} = Q(f, f) \tag{1.1}$$

for a one-particle distribution function $f = f(t, \mathbf{x}, \mathbf{p})$ that depends on the time $t \in \mathbf{R}_+$, the position $\mathbf{x} \in \mathbf{R}^3$, and the momentum $\mathbf{p} \in \mathbf{R}^3$, where $p_0 = (1 + |\mathbf{p}|^2)^{1/2}$ and $Q(f, f)$ is the relativistic collision operator whose structure will be addressed below. Here and throughout

This work was supported by NSFC 10271121 and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, the Ministry of Education of China, and sponsored by joint grants of NSFC 10511120278/10611120371 and RFBR 04-02-39026.

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this paper, $\mathbf{R}_+$ represents the positive side of the real axis including its origin and $\mathbf{R}^3$ denotes a three-dimensional Euclidean space.

The collision operator Q is expressed by the difference between the gain and loss terms respectively defined by

$$Q^+(f,f)(t,\mathbf{x},\mathbf{p}) = \int_{\mathbf{R}^3\times S^2} f(t,\mathbf{x},\mathbf{p}')f(t,\mathbf{x},\mathbf{p}_1')\frac{B(g,\theta)}{p_0p_{10}}d^3\mathbf{p}_1 d\Omega \tag{1.2}$$

and

$$Q^-(f,f)(t,\mathbf{x},\mathbf{p}) = \int_{\mathbf{R}^3\times S^2} f(t,\mathbf{x},\mathbf{p})f(t,\mathbf{x},\mathbf{p}_1)\frac{B(g,\theta)}{p_0p_{10}}d^3\mathbf{p}_1 d\Omega. \tag{1.3}$$

In (1.2) and (1.3), S^2 is a unit sphere surface in $\mathbf{R}^3$, $(\mathbf{p}',\mathbf{p}_1')$ are dimensionless momenta after collision of two particles having precollisional dimensionless momenta $(\mathbf{p},\mathbf{p}_1)$, p_{10} is defined by $p_{10}=(1+|\mathbf{p}_1|^2)^{1/2}$ and represents the dimensionless energy of the colliding particle having the momentum $\mathbf{p}_1$ immediately before collision of two particles, $B(g,\theta)$ is the collision kernel of the momentum distance and scattering angle variables g and θ which are respectively denoted by

$$g=\sqrt{|p_{10}-p_0|^2-|\mathbf{p}_1-\mathbf{p}|^2}/2 \tag{1.4}$$

and

$$\theta = \arccos\{1+[(p_0-p_{10})(p_0-p_0')-(\mathbf{p}-\mathbf{p}_1)(\mathbf{p}-\mathbf{p}')]/(2g^2)\} \tag{1.5}$$

with $p_0'=(1+|\mathbf{p}'|^2)^{1/2}$ representing the dimensionless energy of the colliding particle having the momentum $\mathbf{p}'$ immediately after collision of two particles, and $d\Omega=\sin\theta d\theta d\psi$ is the differential of area on S^2 for any $\theta\in[0,\pi]$ and $\psi\in[0,2\pi]$.

The initial data $f|_{t=0}=f_0(\mathbf{x},\mathbf{p})$ in $\mathbf{R}^3\times\mathbf{R}^3$ are required to satisfy

$$f_0\geq 0 \quad \text{a.e. in } \mathbf{R}^3\times\mathbf{R}^3, \quad \iint_{\mathbf{R}^3\times\mathbf{R}^3} f_0(1+p_0+|\ln f_0|)d^3\mathbf{x}d^3\mathbf{p}<\infty. \tag{1.6}$$

In (1.6), the third term of the integral can control the Boltzmann entropy at an initial time while the two other terms of the integral, from left to right, respectively represent the mass and the energy in the relativistic system at the initial time. The finiteness of all the integrals states that the relativistic system has finite mass, energy and entropy at the initial state.

There are many authors who have contributed to the study of the Cauchy problem for RBE, e.g., Bichteler [3], Bancel [2], Dudyński and Ekiel-Jeżewska [7–10], Glassey and Strauss [12–14], Andréasson [1], Cercignani and Kremer [4], Glassey [11]. Many other relevant papers and books can be found in the references mentioned above.

The DiPerna and Lions techniques (see [6]) for the nonrelativistic Boltzmann equation were originally applied by Dudyński and Ekiel-Jeżewska [10] to their proof of a global existence of solutions to the Cauchy problem for RBE with the assumptions of the relativistic scattering cross section excluding the relativistic hard interactions and the initial data satisfying finite mass, energy and entropy. Unlike in the nonrelativistic case, the relativistic initial data is not required to have a finite "inertia" since the causality of solutions to RBE is used by Dudyński and Ekiel-Jeżewska into their proof. Their results are correct (except the boundness of the entropy at any time without such an assumption as a finite "inertia" considered below) but their assumption of the relativistic scattering cross section does not include the cases of the relativistic hard interactions. After that, a different proof was also given in [18] to show a global existence of solutions to the large-data Cauchy problem for RBE with some relativistic hard interactions. In his proof, the property of the causality is not used directly in solving the Cauchy problem but it is assumed that the initial data satisfies

$$f_0 \geq 0 \quad \text{a.e. in } \mathbf{R}^3 \times \mathbf{R}^3, \ \iint_{\mathbf{R}^3\times\mathbf{R}^3} f_0(1 + p_0|\mathbf{x}|^2 + p_0 + |\ln f_0|)d^3\mathbf{x}d^3\mathbf{p} < \infty, \tag{1.7}$$

i.e., finite mass, "inertia", energy and entropy. Unlike in the nonrelativistic case, the initial condition (1.7) indicates that the relativistic "inertia" is required to involve an integral of $f_0 p_0 |\mathbf{x}|^2$ over space and momentum variables because of the fact that the physically natural *a priori* estimates of the solutions to RBE are made by using the relativistic collision invariant $p_0(\mathbf{x}-\mathbf{p}t/p_0)^2 + t^2/p_0$ of two colliding particles immediately before and after collision while those to the nonrelativistic Boltzmann equation result from the nonrelativistic collision invariant $(\mathbf{x} - \mathbf{vt})^2$.

The objective of this paper is to show that there exists a global mild solutions to the large-data Cauchy problem for RBE with some relativistic hard interactions under the condition of the initial data f_0 satisfying (1.6), that is,

Theorem 1.1 *Let $B(g,\theta)$ be the relativistic collision kernel of RBE* (1.1), *defined above, and B_R a ball with a center at the origin and a radius R, $A(g) = \int_{S^2} B(g,\theta)d\Omega$. Assume that*

$$B(g,\theta) \geq 0 \quad \textit{a.e. in } \ [0,+\infty) \times S^2, B(g,\theta) \in L^1_{loc}(\mathbf{R}^3 \times S^2), \tag{1.8}$$

$$\frac{1}{p_0^2}\int_{B_R} \frac{A(g)}{p_{10}} d^3\mathbf{p}_1 \to 0 \quad \textit{as } |\mathbf{p}| \to +\infty, \textit{ for all } R \in (0,+\infty). \tag{1.9}$$

Then RBE (1.1) *has a mild or equivalently a renormalized solution f through initial data f_0 with* (1.6), *satisfying the following properties*

$$f \in C([0,+\infty); L^1(\mathbf{R}^3 \times \mathbf{R}^3)), \tag{1.10}$$

$$L(f) \in L^\infty([0,+\infty); L^1(\mathbf{R}^3 \times B_R)), \quad \textit{for all } R \in (0,+\infty), \tag{1.11}$$

$$\frac{Q^+(f,f)}{1+f} \in L^1([0,T]; L^1(\mathbf{R}^3 \times B_R)), \quad \textit{for all } R, T \in (0,+\infty), \tag{1.12}$$

$$\sup_{t\geq 0} \iint_{\mathbf{R}^3\times\mathbf{R}^3} f(1 + p_0 + \ln f)d^3\mathbf{x}d^3\mathbf{p} < +\infty. \tag{1.13}$$

This theorem is in fact an extension of the result given by Dudyński and Ekiel-Jeżewska [10] to the relativistic system with hard interactions. The reason is found that both the causality of RBE and the conservation of mass and energy in the relativistic system guarantee the relativistic "inertia" involving an integral of $f p_0 |\mathbf{x}|^2$ over all the space and momentum variables to be successfully estimated at any time.

It is clear that the condition (1.8) is equivalent to the following one:

$$\sigma(g,\theta) \geq 0 \quad \text{a.e. in } [0,+\infty) \times S^2, \ g(1+g^2)^{1/2}\sigma(g,\theta) \in L^1_{loc}([0,+\infty) \times S^2), \tag{1.14}$$

which was first defined by Jiang [18]. The assumption (1.9) was originally introduced by Jiang (see [17], [19]). Obviously, the relativistic assumptions (1.8) and (1.9) are similar to the following nonrelativistic ones adopted by DiPerna and Lions [6]:

$$B(\mathbf{z},\omega)\geq 0 \quad \text{a.e. in } \mathbf{R}^N\times S^{N-1},\ B(\mathbf{z},\omega)\in L^1_{loc}(\mathbf{R}^N\times S^{N-1}), \tag{1.15}$$

$$\frac{1}{1+|\xi|^2}\int_{|\mathbf{z}-\xi|\leq R}\tilde{A}(\mathbf{z})d\mathbf{z}\to 0 \quad \text{as } |\xi|\to+\infty,\ \text{for all } R\in(0,+\infty), \tag{1.16}$$

where $B(\mathbf{z},\omega)$ is a function of $|\mathbf{z}|$, $|(\mathbf{z},\omega)|$ only, $\tilde{A}(\mathbf{z})=\int_{S^{N-1}}B(\mathbf{z},\omega)d\omega$. It is also easy to see that the condition (1.9) includes some relativistic hard interactions defined as $\int_{S^2}B(g,\theta)d\Omega\geq Cg^2$, where C is a positive constant (see [9]). For example, if $B(g,\theta)=s^{\frac{1}{2}}g^{\beta+1}\sin^\gamma(\theta)$ where $\gamma>-2$, $0\leq\beta<\min(2,2+\gamma)$, then $B(g,\theta)$ satisfies (1.8) and (1.9), and it is a relativistic hard interaction kernel. But it was assumed by Dudyński and Ekiel-Jeżewska (see [10]) that $B(g,\theta)$ satisfies (1.8) and the following condition:

$$\frac{1}{p_0}\int_{B_R}\frac{A(g)}{p_{10}}d^3\mathbf{p}_1\to 0 \quad \text{as } |\mathbf{p}|\to+\infty,\ \text{for all } R\in(0,+\infty), \tag{1.17}$$

where B_R and $A(g)$ are the same as (1.9); it has been claimed in [10] that their assumptions of $B(g,\theta)$ exclude the relativistic hard interactions. In fact, since $g^2=(p_{10}p_0-\mathbf{p}_1\mathbf{p}-1)/2$, it is easy to see that

$$\int_{B_R}\frac{A(g)}{p_{10}}d^3\mathbf{p}_1\geq 2\pi C[p_0R^3/3-R\sqrt{1+R^2}/2+\ln(R+\sqrt{1+R^2})/2]$$

for $A(g)\geq Cg^2$, where C is a positive constant and $R>0$. This implies that (1.17) does not hold in the relativistic hard interaction cases. It follows that (1.17) is more restrictive than (1.9).

The rest of this paper is organized as follows. Besides the conservation laws of mass, momenta and energy in the relativistic system, the property that the entropy of the system is always a nondecreasing function of t is described in Sect. 2. Finally, in Sect. 3, the DiPerna and Lions techniques and the Dudyński and Ekiel-Jeżewska devices are successfully applied to prove the global existence of solutions to the Cauchy problem for RBE with hard interactions in L^1 if the initial data satisfies finite mass, energy and entropy. The physically natural *a priori* estimates of the solutions are also shown to be bounded in any given finite time interval.

2 Conservation Laws and Entropy

As in the nonrelativistic case, the structure of the relativistic collision operator maintains not only the conversation of mass, momenta and energy in the relativistic system, but also the property that the entropy of the system does not decrease.

Since energy and momenta of two colliding particles conserve before and after collision, that is, $\mathbf{p}+\mathbf{p}_1=\mathbf{p}'+\mathbf{p}'_1$, $p_0+p_{10}=p'_0+p'_{10}$, it is easily proved that

$$s=s',\qquad g=g', \tag{2.1}$$

where $s' = |p'_{10} + p'_0|^2 - |\mathbf{p}'_1 + \mathbf{p}'|^2$, $g' = \sqrt{|p'_{10} - p'_0|^2 - |\mathbf{p}'_1 - \mathbf{p}'|^2}/2$. It is also shown that

$$\cos\theta = 1 - 2[|p_0 - p'_0|^2 - |\mathbf{p} - \mathbf{p}'|^2]/(s-4). \tag{2.2}$$

It requires further analysis of the relativistic collision term to show the conversation laws in the relativistic system. By using (2.1) and (2.2), it can be easily proved that

$$\int_{\mathbf{R}^3} \psi(\mathbf{p}) Q(\varphi,\varphi) d^3\mathbf{p} = \frac{1}{4} \iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{d^3\mathbf{p}_1}{p_0 p_{10}} \int_{S^2} d\Omega\, B(g,\theta)[\varphi(\mathbf{p}')\varphi(\mathbf{p}'_1) - \varphi(\mathbf{p})\varphi(\mathbf{p}_1)] \\ \times [\psi(\mathbf{p}) + \psi(\mathbf{p}_1) - \psi(\mathbf{p}') - \psi(\mathbf{p}'_1)] \tag{2.3}$$

if $Q^{\pm}(\varphi,\varphi)\psi(\mathbf{p}) \in L^1(\mathbf{R}^3)$ for any given $\psi(\mathbf{p}) \in L^\infty(\mathbf{R}^3)$ and every given $\varphi(\mathbf{p}) \in L^1(\mathbf{R}^3)$. It follows from (2.3) that $\int_{\mathbf{R}^3} \bar{\psi} Q(f,f) d^3\mathbf{p} = 0$ if $f = f(t,\mathbf{x},\mathbf{p})$ is a distributional solution to RBE (1.1) such that $\int_{\mathbf{R}^3} \bar{\psi} Q(f,f) d^3\mathbf{p} < +\infty$ for almost all t and $\mathbf{x}$ and $\bar{\psi} = \bar{b}_0 + \mathbf{bp} + c_0 p_0$, where $\bar{b}_0 \in \mathbf{R}$, $\mathbf{b} \in \mathbf{R}^3$, $c_0 \in \mathbf{R}$. Furthermore, it is at least formally found that $\iint_{\mathbf{R}^3\times\mathbf{R}^3} \bar{\psi} f d^3\mathbf{x} d^3\mathbf{p}$ is independent of t for any distributional solution f to RBE (1.1). This yields the conservation of mass, momentum and kinetic energy of the relativistic system.

It is well known that the nonrelativistic Boltzmann equation has the conservation of the integral of $f(\mathbf{x} - \mathbf{v}t)^2$ over all the space and velocity variables besides the conservation of mass, momentum and kinetic energy of the nonrelativistic system. This is because $(\mathbf{x} - \mathbf{v}t)^2$ is an invariant of two nonrelativistic colliding particles immediately before and after collision. In the relativistic case, although $p_0(\mathbf{x} - t\mathbf{p}/p_0)^2 + t^2/p_0$ is a relativistic invariant of two colliding particles immediately before and after collision, the integral of $f[p_0(\mathbf{x} - t\mathbf{p}/p_0)^2 + t^2/p_0]$ over all the space and momentum variables changes with t. In fact, by multiplying RBE (1.1) by $p_0(\mathbf{x} - t\mathbf{p}/p_0)^2 + t^2/p_0$ and integrating by parts over $\mathbf{x}$ and $\mathbf{p}$, it is easy to see that

$$\frac{d}{dt} \iint_{\mathbf{R}^3\times\mathbf{R}^3} f[p_0(\mathbf{x} - t\mathbf{p}/p_0)^2 + t^2/p_0] d^3\mathbf{x} d^3\mathbf{p} \\ = \iint_{\mathbf{R}^3\times\mathbf{R}^3} f\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{p_0}\frac{\partial}{\partial \mathbf{x}}\right)[p_0(\mathbf{x} - t\mathbf{p}/p_0)^2 + t^2/p_0] d^3\mathbf{x} d^3\mathbf{p} \tag{2.4}$$

and hence

$$\frac{d}{dt} \iint_{\mathbf{R}^3\times\mathbf{R}^3} f[p_0(\mathbf{x} - t\mathbf{p}/p_0)^2 + t^2/p_0] d^3\mathbf{x} d^3\mathbf{p} = 2t \iint_{\mathbf{R}^3\times\mathbf{R}^3} f/p_0 d^3\mathbf{x} d^3 \tag{2.5}$$

which yields the estimate of the integral $\iint_{\mathbf{R}^3\times\mathbf{R}^3} f[p_0(\mathbf{x} - t\mathbf{p}/p_0)^2 + t^2/p_0] d^3\mathbf{x} d^3\mathbf{p}$ under the assumption of (1.7). This is why the assumption (1.7) was really made by Jiang [18] before. Fortunately, it can be easily known from (2.5) that

$$\sup_{0\le t\le T} \iint_{\mathbf{R}^3\times\mathbf{R}^3} f p_0(\mathbf{x} - t\mathbf{p}/p_0)^2 d^3\mathbf{x} d^3\mathbf{p} \le \int_{\mathbf{R}^3\times\mathbf{R}^3} f_0(p_0|\mathbf{x}|^2 + T^2) d^3\mathbf{x} d^3\mathbf{p}, \tag{2.6}$$

which is very useful to the estimate of the relativistic entropy integral considered below.

By (2.6), the desired estimate of $\iint_{\mathbf{R}^3\times\mathbf{R}^3} f p_0 |\mathbf{x}|^2 d^3\mathbf{x} d^3\mathbf{p}$ under the assumption of (1.7) can be also made successfully. To show this estimate, it requires the following identity

$$\frac{d}{dt} \iint_{\mathbf{R}^3\times\mathbf{R}^3} f p_0 |\mathbf{x}|^2 d^3\mathbf{x} d^3\mathbf{p} = 2 \iint_{\mathbf{R}^3\times\mathbf{R}^3} f \mathbf{xp} d^3\mathbf{x} d^3\mathbf{p} \tag{2.7}$$

derived by multiplying RBE (1.1) by $p_0|\mathbf{x}|^2$ and integrating by parts over $\mathbf{x}$ and $\mathbf{p}$, and hence

$$\frac{d}{dt}\iint_{\mathbf{R}^3\times\mathbf{R}^3} f p_0|\mathbf{x}|^2 d^3\mathbf{x}d^3\mathbf{p} \le \iint_{\mathbf{R}^3\times\mathbf{R}^3} f p_0|\mathbf{x}|^2 d^3\mathbf{x}d^3\mathbf{p} + \iint_{\mathbf{R}^3\times\mathbf{R}^3} f p_0 d^3\mathbf{x}d^3\mathbf{p}, \tag{2.8}$$

which yields the following inequality

$$\sup_{0\le t\le T}\iint_{\mathbf{R}^3\times\mathbf{R}^3} f p_0|\mathbf{x}|^2 d^3\mathbf{x}d^3\mathbf{p} \le e^T \iint_{\mathbf{R}^3\times\mathbf{R}^3} f_0 p_0(1+|\mathbf{x}|^2) d^3\mathbf{x}d^3\mathbf{p} \tag{2.9}$$

for any given $T > 0$ by multiplying the two sides of (2.8) by e^{-t} and using the conservation of the mass of the relativistic system. The inequality given by (2.9) illustrates that the relativistic "inertia" of $fp_0|\mathbf{x}|^2$ over all the space and momentum variables is at any time controlled by both mass and "inertia" at the initial state of the relativistic system. The above analysis also dedicates that the conservation of mass and energy guarantees the relativistic "inertia" involving an integral of $fp_0|\mathbf{x}|^2$ over all the space and momentum variables to be successfully estimated in the relativistic system at any time.

The physically natural estimates of solutions to RBE (1.1) require not only the relativistic conservation laws but also the property that the entropy is always a nondecreasing function of t in the relativistic system. To show this property of the relativistic entropy, the relativistic entropy identity has to be first considered as in the nonrelativistic case. It is easy to at least formally deduce the following entropy identity

$$\begin{aligned}&\frac{d}{dt}\iint_{\mathbf{R}^3\times\mathbf{R}^3} f\ln f d^3\mathbf{x}d^3\mathbf{p} + \frac{1}{4p_0}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{d^3\mathbf{p}_1}{p_{10}}\int_{S^2} d\Omega B(g,\theta)\\&\quad\times[f(t,\mathbf{x},\mathbf{p}')f(t,\mathbf{x},\mathbf{p}_1') - f(t,\mathbf{x},\mathbf{p})f(t,\mathbf{x},\mathbf{p}_1)]\ln\left[\frac{f(t,\mathbf{x},\mathbf{p}')f(t,\mathbf{x},\mathbf{p}_1')}{f(t,\mathbf{x},\mathbf{p})f(t,\mathbf{x},\mathbf{p}_1)}\right] = 0\end{aligned} \tag{2.10}$$

by multiplying RBE (1.1) by $1 + \ln f$, integrating over $\mathbf{x}$ and $\mathbf{p}$ and using (2.3). In general, for convenience, put $H(t) = \iint_{\mathbf{R}^3\times\mathbf{R}^3} f\ln f d^3\mathbf{x}d^3\mathbf{p}$, and $H(t)$ is called H-function. Boltzmann's entropy is usually defined by $-H(t)$. The second term in (2.10) is nonnegative and so $H(t)$ is a nonincreasing function of t. This means that the entropy of the relativistic system does not decrease. This property allows the desired estimate of the relativistic entropy to be derived from the Cauchy problem for RBE.

In fact, the entropy can be controlled by the integral $\iint_{\mathbf{R}^3\times\mathbf{R}^3} f|\ln f|d^3\mathbf{x}d^3\mathbf{p}$ for any nonnegative solution to RBE (1.1) and so it is natural to make the considered estimate of the integral instead of the entropy. Notice that

$$\begin{aligned}&\iint_{\mathbf{R}^3\times\mathbf{R}^3} f|\ln f|d^3\mathbf{x}d^3\mathbf{p}\\&\quad= \iint_{\mathbf{R}^3\times\mathbf{R}^3} f\ln f d^3\mathbf{x}d^3\mathbf{p} - 2\iint_{f\le 1} f|\ln f|d^3\mathbf{x}d^3\mathbf{p}\\&\quad\le \iint_{\mathbf{R}^3\times\mathbf{R}^3} f\ln f d^3\mathbf{x}d^3\mathbf{p} + 2\iint_{\mathbf{R}^3\times\mathbf{R}^3} f[p_0(\mathbf{x}-t\mathbf{p}/p_0)^2 + p_0]d^3\mathbf{x}d^3\mathbf{p}\\&\qquad + 2\iint_{f\le\exp(-|\mathbf{x}-t\mathbf{p}/p_0|^2-p_0)} f\ln(1/f)d^3\mathbf{x}d^3\mathbf{p}\end{aligned}$$

$$\leq \iint_{\mathbf{R}^3\times\mathbf{R}^3} f \ln f d^3\mathbf{x} d^3\mathbf{p} \\ + 2\iint_{\mathbf{R}^3\times\mathbf{R}^3} f[p_0(\mathbf{x}-t\mathbf{p}/p_0)^2 + p_0] d^3\mathbf{x} d^3\mathbf{p} + C_1 \quad (2.11)$$

where C_1 is some positive constant independent of f. By using (2.6), (2.10) and (2.11), it can be deduced that

$$\sup_{0\leq t\leq T}\left[\iint_{\mathbf{R}^3\times\mathbf{R}^3} f|\ln f| d^3\mathbf{x} d^3\mathbf{p}\right] \\ \leq \iint_{\mathbf{R}^3\times\mathbf{R}^3} f_0[2T^2 + 2p_0(1+|\mathbf{x}|^2) + |\ln f_0|] d^3\mathbf{x} d^3\mathbf{p} + C_1. \quad (2.12)$$

This implies that the boundness of the entropy at any time might not be guaranteed without such an assumption as the finite initial "inertia" mentioned above.

It is worth mentioning that many other properties of RBE (1.1) can be found from the book of Cercignani and Kremer [4].

3 Proof of Global Existence

In order to prove Theorem 1.1, both the collision kernel and the initial data have to be first truncated and regularized by using the same approximation scheme as given by DiPerna and Lions [6] in the nonrelativistic case. The collision kernel $B(g,\theta)$ of RBE (1.1) can be truncated to obtain $B_n(g,\theta) \in L^\infty \cap L^1(\mathbf{R}^3; L^1(S^2))$ such that

$$\iint_{B_R\times S^2} d^3\mathbf{p} d\Omega |B_n(g,\theta) - B(g,\theta)| \to 0 \quad (3.1)$$

uniformly in $\{\mathbf{p}_1 : |\mathbf{p}_1| \leq k\}$ as $n \to +\infty$ for all $R, k \in (0,+\infty)$. Then it leads to the problem of solving the approximate equation

$$\frac{\partial f^n}{\partial t} + \frac{\mathbf{p}}{p_0}\frac{\partial f^n}{\partial \mathbf{x}} = \tilde{Q}_n(f^n, f^n) \text{ in } (0,\infty)\times\mathbf{R}^3\times\mathbf{R}^3. \quad (3.2)$$

Here and below, $\tilde{Q}_n$ is defined by $\tilde{Q}_n(\varphi,\varphi) = (1+\frac{1}{n}\int_{\mathbf{R}^3}|\varphi| d^3\mathbf{p})^{-1} Q_n(\varphi,\varphi)$ and

$$Q_n(\varphi,\varphi) = \frac{1}{p_0}\int_{\mathbf{R}^3}\frac{d^3\mathbf{p}_1}{p_{10}}\int_{S^2} d\Omega[\varphi(\mathbf{p}')\varphi(\mathbf{p}_1') - \varphi(\mathbf{p})\varphi(\mathbf{p}_1)]B_n(g,\theta). \quad (3.3)$$

It follows from (3.3) that for all $\varphi, \psi \in L^\infty([0,+\infty)\times\mathbf{R}^3\times\mathbf{R}^3) \cap L^1(\mathbf{R}^3\times\mathbf{R}^3)$,

$$\|\tilde{Q}_n(\varphi,\varphi)\|_{L^\infty([0,+\infty)\times\mathbf{R}^3\times\mathbf{R}^3)} \leq C_n\|\varphi\|_{L^\infty([0,+\infty)\times\mathbf{R}^3\times\mathbf{R}^3)}, \quad (3.4)$$

$$\|\tilde{Q}_n(\varphi,\varphi)\|_{L^1(\mathbf{R}^3\times\mathbf{R}^3)} \leq C_n\|\varphi\|_{L^1(\mathbf{R}^3\times\mathbf{R}^3)}, \quad (3.5)$$

$$\|\tilde{Q}_n(\varphi,\varphi) - \tilde{Q}_n(\psi,\psi)\|_{L^1(\mathbf{R}^3\times\mathbf{R}^3)} \leq C_n\|\varphi-\psi\|_{L^1(\mathbf{R}^3\times\mathbf{R}^3)}, \quad (3.6)$$

here and below everywhere, C_n is a nonnegative constant independent of φ and ψ.

By following DiPerna and Lions [6], the initial data f_0 can be first truncated and regularized to get a sequence of nonnegative functions $f_0^n \in D(\mathbf{R}^3 \times \mathbf{R}^3)$ such that

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3} d^3\mathbf{x} d^3\mathbf{p} |f_0 - f_0^n|(1 + p_0|\mathbf{x}|^2 + p_0) \to 0 \quad \text{as } n \to +\infty, \tag{3.7}$$

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3} d^3\mathbf{x} d^3\mathbf{p} f_0^n |\ln f_0^n| \le C \quad \text{independent of } n. \tag{3.8}$$

Then there exists a unique nonnegative distributional solution $f_m^n = f_m^n(t, \mathbf{x}, \mathbf{p})$ to the problem of the approximate equation (3.2) with the initial data $f_{m,0}^n \equiv f_0^n 1_{B_m}(\mathbf{x})$ for any given ball $B_m \equiv \{\mathbf{x} : |\mathbf{x}| < m\}$. It can be also easily proved that $\tilde{Q}_n(f_m^n, f_m^n) \in L^1_{loc}(\mathbf{R}^3 \times \mathbf{R}^3)$ and that f_m^n satisfies the following properties:

$$0 \le f_m^n \in L^\infty \cap L^1((0,T)\times\mathbf{R}^3\times\mathbf{R}^3) \quad (\forall T < +\infty), \tag{3.9}$$

$$f_m^n(t, \mathbf{x}, \mathbf{p}) \in C([0,+\infty); L^1(\mathbf{R}^3\times\mathbf{R}^3)). \tag{3.10}$$

Let $\tilde{L}_n$ be denoted by

$$\tilde{L}_n(\varphi) = \left(1 + \frac{1}{n}\int_{\mathbf{R}^3} |\varphi| d^3\mathbf{p}\right)^{-1} \frac{1}{p_0} \int_{\mathbf{R}^3} \frac{d^3\mathbf{p}_1}{p_{10}} \int_{S^2} d\Omega \varphi(\mathbf{p}_1) B_n(g,\theta). \tag{3.11}$$

Put $\tilde{Q}_n^-(\varphi,\varphi) = \varphi(\mathbf{p})\tilde{L}_n(\varphi)$ and $\tilde{Q}_n^+(\varphi,\varphi) = \tilde{Q}_n(\varphi,\varphi) - \tilde{Q}_n^-(\varphi,\varphi)$. It is then obvious to see that

$$\tilde{Q}_n^+(f_m^n, f_m^n),\ \tilde{Q}_n^-(f_m^n, f_m^n) \in L^1_{loc}((0,+\infty)\times\mathbf{R}^3\times\mathbf{R}^3). \tag{3.12}$$

By using (2.9) and (2.12) and with the help of Gronwall's inequality, it can be further found that

$$\sup_{t\ge 0} \iint_{\mathbf{R}^3\times\mathbf{R}^3} f_m^n(1 + p_0 + \ln f_m^n) d^3\mathbf{x} d^3\mathbf{p} \le C_0, \tag{3.13}$$

$$\sup_{0\le t\le T} \iint_{\mathbf{R}^3\times\mathbf{R}^3} f_m^n(p_0|\mathbf{x}|^2 + |\ln f_m^n|) d^3\mathbf{x} d^3\mathbf{p} \le C_{Tm}. \tag{3.14}$$

It also follows by (2.10) that

$$\begin{aligned} &\frac{1}{4}\int_0^{+\infty}\int_{\mathbf{R}^3}\left\{\left(1+\int_{\mathbf{R}^3} f_m^n d^3\mathbf{p}\right)^{-1} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times S^2} \frac{d^3\mathbf{p}_1 d^3\mathbf{p} d\Omega}{p_{10}p_0} B_n(g,\theta)\right. \\ &\quad \left.\times (f_m^{n\prime} f_{m1}^{n\prime} - f_m^n f_{m1}^n)\ln\left(\frac{f_m^{n\prime} f_{m1}^{n\prime}}{f_m^n f_{m1}^n}\right)\right\} d\sigma d^3\mathbf{x} \le C_{Tm}. \end{aligned} \tag{3.15}$$

In (3.13), C_0 is a positive constant which is only dependent of f_0. In (3.14) and (3.15), C_{Tm} is a positive constant dependent of m, f_0 and T except of n. It can be easily deduced from (3.13) and (3.14) that $\{f_m^n\}_{n=1}^\infty$ is weakly compact in $L^1((0,T)\times\mathbf{R}^3\times\mathbf{R}^3)$ for $T \in (0,+\infty)$ when m is fixed. Thus it may be assumed without loss of generality that f_m^n converges weakly in $L^1((0,T)\times\mathbf{R}^3\times\mathbf{R}^3)$ to $f_m \in L^1_{loc}([0,+\infty)\times\mathbf{R}^3\times\mathbf{R}^3)$ as $n \to +\infty$ for all $T < +\infty$ when m is fixed. Obviously, $f_m \ge 0$ and $f_m|_{t=0} = f_{m,0}$ for almost every $(\mathbf{x},\mathbf{p}) \in \mathbf{R}^3\times\mathbf{R}^3$ where $f_{m,0} \equiv f_0 1_{B_m}(\mathbf{x})$.

It can be also proven that for any fixed $m, T, R \in (0, +\infty)$, $\{\tilde{Q}_n^{\pm}(f_m^n, f_m^n)/(1+f_m^n)\}_{n=1}^{\infty}$ are weakly compact subsets of $L^1((0,T)\times\mathbf{R}^3\times B_R)$. It further follows that f_m is a global mind solution to RBE (1.1) with the initial data $f_{m,0}$, satisfying

$$f_m \in C([0,+\infty); L^1(\mathbf{R}^3\times\mathbf{R}^3)), \tag{3.16}$$

$$L(f_m) \in L^\infty([0,+\infty); L^1(\mathbf{R}^3\times B_R)), \quad \text{for all } R\in(0,+\infty), \tag{3.17}$$

$$\frac{Q^+(f_m,f_m)}{1+f_m} \in L^1([0,T]; L^1(\mathbf{R}^3\times B_R)), \quad \text{for all } R, T\in(0,+\infty), \tag{3.18}$$

$$\sup_{m\geq 1, t\geq 0} \iint_{\mathbf{R}^3\times\mathbf{R}^3} f_m(1+p_0+\ln f_m)d^3\mathbf{x}d^3\mathbf{p} < +\infty, \tag{3.19}$$

by analyzing step by step the relaxation of the normalization and construction of subsolutions and supersolutions with a similar device to that given by DiPerna and Lions [6]. This analysis not only allows for the relations among three different types of solutions to RBE (1.1) (see [18]) but also requires the momentum-averaged compactness of the transport operator of RBE (1.1) (see [15] or [16]). Here, (3.19) is derived from (3.13).

Below is a modification of the devices of Dudyński and Ekiel-Jeżewska [10]. By using both the causality and the uniqueness of solution to the approximate relativistic Boltzmann equation (3.2), it is easy to see that if n is fixed, f_m^n is convergent as $m \to \infty$ for almost every $(t, \mathbf{x}, \mathbf{p})$. Put $f^n = \lim_{m\to\infty} f_m^n$. Then f^n is a unique global distributional solution to the approximate equation (3.2) through f_0^n. It can be also found that $\{f^n\}_{n=1}^{\infty}$ is weakly compact in $L^1((0,T)\times B_m\times\mathbf{R^3})$ for any given $T>0$ and $m>0$. It may be assumed without loss of generality that f^n converges weakly in $L^1((0,T)\times\mathbf{R^3}\times\mathbf{R^3})$ to f for any given $T>0$. It follows that f_m converges to f as $m\to\infty$ for almost every $(t,\mathbf{x},\mathbf{p})$. Hence f is a global mild solution to RBE (1.1) through f_0. By (3.16), (3.17), (3.18) and (3.19), it can be also shown that f satisfies (1.10), (1.11), (1.12) and (1.13). This completes the proof of Theorem 1.1.

Remark 1 The content of this paper advances that contained in references [16–19]. One advantage is to employ the core new estimates (2.9) and (2.12) to obtain a unique nonnegative distributional solution f_m^n to the problem of the approximate equation (3.2) with a class of initial data which is more natural than the ones considered previously by Dudyński and Ekiel-Jeżewska. Another is to use the assumptions (1.8) and (1.9) of the relativistic collision kernel with some relativistic hard interactions to show that the Cauchy problem of RBE (1.1) has a global mild solution on the condition of the finite initial physically natural bounds excluding the finite initial "inertia".

Acknowledgements The author would like to thank the referees of this paper for their valuable comments and suggestions.

References

1. Andréasson, H.: Regularity of the gain term and strong L^1 convergence to equilibrium for the relativistic Boltzmann equation. SIAM J. Math. Anal. **27**, 1386–1405 (1996)
2. Bancel, D.: Problème de Cauchy pour l'équation de Boltzmann en relativité générale. Ann. Inst. Henri Poincaré **18**, 263–284 (1973)

3. Bichteler, K.: On the Cauchy problem of the relativistic Boltzmann equation. Commun. Math. Phys. **4**, 352–364 (1967)
4. Cercignani, C., Kremer, G.: The Relativistic Boltzmann Equation, Theory and Applications. Birkhaeuser, Boston (2002)
5. de Groot, S.R., Van Leeuwen, W.A., Van Weert, Ch.G.: Relativistic Kinetic Theory. North-Holland, Amsterdam (1980)
6. DiPerna, R.J., Lions, P.L.: On the Cauchy problem for Boltzmann equations: global existence and weak stability. Ann. Math. **130**, 321–366 (1989)
7. Dudyński, M., Ekiel-Jeżewska, M.L.: Causality of the linearized relativistic Boltzmann equation. Phys. Rev. Lett. **55**, 2831–2834 (1985)
8. Dudyński, M., Ekiel-Jeżewska, M.L.: Errata: causality of the linearized relativistic Boltzmann equation. Phys. Rev. Lett. **56**, 2228–2228 (1985)
9. Dudyński, M., Ekiel-Jeżewska, M.L.: On the linearized relativistic Boltzmann equation. Commun. Math. Phys. **115**, 607–629 (1988)
10. Dudyński, M., Ekiel-Jeżewska, M.L.: Global existence proof for relativistic Boltzmann equation. J. Stat. Phys. **66**(3/4), 991–1001 (1992)
11. Glassey, R.: Global solutions to the Cauchy problem for the relativistic Boltzmann equation with near-vacuum data. Commun. Math. Phys. **264**, 705–724 (2006)
12. Glassey, R., Strauss, W.: On the derivatives of the collision map of relativistic particles. Transp. Theory Stat. Phys. **20**, 55–68 (1991)
13. Glassey, R., Strauss, W.: Asymptotic stability of the relativistic equilibrium. Publ. RIMS Kyoto Univ. **29**, 301–347 (1992)
14. Glassey, R., Strauss, W.: Asymptotic stability of the relativistic Maxwellian via fourteen moments. Transp. Theory Stat. Phys. **24**, 657–678 (1995)
15. Golse, F., Lions, P.L., Perthame, B., Sentis, R.: Regularity of the moments of the solution of a transport equation. J. Functional Anal. **76**, 110–125 (1988)
16. Jiang, Z.: Compactness related to the transport operator of the relativistic Boltzmann equation. Acta Math. Sci. **17**(3), 330–335 (1997) (in Chinese)
17. Jiang, Z.: A proof of the existence of global solutions for the relativistic Boltzmann equation with hard interactions. J. Wuhan Univ. Natur. Sci. Ed. **43**(1), 1–6 (1997) (in Chinese)
18. Jiang, Z.: On the relativistic Boltzmann equation. Acta Math. Sci. **18**(3), 348–360 (1998)
19. Jiang, Z.: On the Cauchy problem for the relativistic Boltzmann equation in a periodic box: global existence. Transp. Theory Stat. Phys. **28**(6), 617–628 (1999)